ALTERNATIVE ANALYTICAL MEAN-SQUARE CALCULATIONS FOR DYNAMIC SYSTEMS SUBJECT TO RANDOM INPUTS

Robin C. Redfield
Department of Engineering Mechanics
United States Air Force Academy
Colorado

ABSTRACT

Output variables of linear dynamic systems subject to random inputs are often quantified by mean square calculations. Computationally, these involve integration of the frequency response magnitude squared over all frequency. Numerically, this is an easy task and analytically, methods exist to find mean square values as functions of transfer function (frequency response) coefficients. This paper develops further analytical techniques to calculate mean-square values as functions of system pole-zero locations and as functions of eigen-properties and system matrices. These other analytical representations may provide paths to further insight into dynamic system response and optimal design/tuning of dynamic systems.

INTRODUCTION

Vibratory systems often are excited by inputs that can be approximated as "white" in frequency content. In other cases, white noise inputs processed through linear filters approximate inputs or disturbances to such systems. Choosing design parameters and/or feedback control gains for such systems can entail examining mean-square responses of key performance variables and tuning for some optimum or allowable trade-offs.

For example, ground vehicle suspensions excited by roadway unevenness are typically tuned for body acceleration or jerk (weighted over frequency perhaps), suspension travel, road-tire contact force, and other performances. Parameters such as spring and shock coefficients, tire stiffness and unsprung mass, and active control gains can be optimized for performance including required control power.

Calculating these average responses for linear systems can always be done numerically since they involve integrating under the squared magnitude of a frequency response function. With known techniques, these responses can also be found analytically using frequency response coefficients. The advantage of the analytical techniques is the possible insight gained and the global relationships derived between parameters and response.

The purpose of this work is to develop further analytical relationships between system poles and zeros, eigen-properties, and dynamic system matrices; and mean-square performance. The hope is that these relationships will give increased understanding into dynamic system behavior and tuning of optimal systems.

The paper is in three main sections. The first reviews the relationship between transfer function coefficients and mean-square response. The second section relates pole and zero locations to response, and the third section develops mean-square response in terms of eigen-properties and system matrices.

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MEAN-SQUARE RESPONSE TO WHITE NOISE INPUTS

Spectral density theory [1] shows that the mean-square output of a linear system, $E[y^2]$[1], is the following function of the spectral density of the input and the frequency response magnitude squared[2].

$$E[y^2] = \int_{-\infty}^{+\infty} |G(iw)|^2 S_U(w)dw \qquad (1)$$

$G(s)|_{s=iw}$ is frequency response, y is output, U is input and $S_U(w)$ is the spectral density of the input. If the input spectral density can be approximated as white noise so that $S_U(w)=S_O$ (a constant), only the integral $\int_{-\infty}^{+\infty} |G(iw)|^2 dw$ requires determination. The following sections detail three approaches to the analytical calculation of this magnitude squared integral.

E[y²] AS A FUNCTION OF TRANSFER FUNCTION COEFFICIENTS

Since frequency response (transfer function) coefficients are typically functions of dynamic system parameters, it can be very useful to calculate $\int_{-\infty}^{+\infty} |G(iw)|^2 dw$ as a function of these coefficients. James et. al. [2] developed these functions for up to order seven frequency response functions and they have served good use to this day. Both [1] and [2] have these functions in appendices.

For example, given

$$G(iw) = \frac{B_0}{A_0 + iwA_1} \qquad (2)$$

then $$\int_{-\infty}^{+\infty} |G(iw)|^2 dw = \frac{pB_0^{\,2}}{A_0 A_1} \qquad (3)$$

And given

$$G(iw) = \frac{B_0 + iwB_1}{A_0 + iwA_1 - w^2 A_2} \qquad (4)$$

then $$\int_{-\infty}^{+\infty} |G(iw)|^2 dw = \frac{p(A_0 B_1^{\,2} + A_2 B_0^{\,2})}{A_0 A_1 A_2} \qquad (5)$$

Functions such as these have been used by Redfield and Karnopp [3], Redfield [4], and Karnopp [5] to calculate vehicle suspension mean-square response to a random roadway input and for determining optimum tuning of suspensions with trade-offs between ride, suspension travel, and tire contact forces. The following sections calculate these integrals with alternative methods.

E[y²] AS A FUNCTION OF POLES AND ZEROS

Development

This section develops the relationship between pole-zero locations and mean-square response. These relationships may facilitate further understanding of dynamic system behavior and optimal tuning. The thrust of this work is to develop the functions and leave significant interpretation to future study.

We will write the transfer function, $G(s)$, as a rational function with a gain, k, factored zeros, z_j, and factored eigenvalues, l_i (poles) in equation 6.

$$G(s) = k\frac{\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - l_i)} \qquad (6)$$

The frequency response, $G(w)$, is found by letting $s \rightarrow iw$.

$$G(iw) = k\frac{\prod_{j=1}^{m}(iw - z_j)}{\prod_{i=1}^{n}(iw - l_i)} \qquad (7)$$

The magnitude squared of the transfer function is the product of $G(iw)$ with its complex conjugate $G^*(iw)$ and can be calculated as shown in equation 8.

$$|G(iw)|^2 = G(iw)G^*(iw) = G(iw)G(-iw)\ ^3 \qquad (8)$$

From equation 7, magnitude squared becomes

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[1] E[•] is the expectation operator and y is the output variable.

[2] |•| denotes magnitude.

[3] Because all complex factors arise as complex conjugates, the conjugate can be calculated with the negative argument.

$$|G(iw)|^2 = k^2 \frac{\prod_{j=1}^{m}(w^2 + z_j^{\,2})}{\prod_{i=1}^{n}(w^2 + l_i^{\,2})} \tag{9}$$

This can be written in factored form ready for integration by the theory of residues [1] in Equation 10.

$$|G(iw)|^2 = k^2 \frac{\prod_{j=1}^{m}(w + iz_j)(w - iz_j)}{\prod_{i=1}^{n}(w + il_i)(w - il_i)} \tag{10}$$

From residue theory with a contour around the closed, upper-half ω-plane, the integral of the magnitude squared over all frequency is

$$\int_{-\infty}^{+\infty} |G(iw)|^2 dw = 2pi \sum C_j \tag{11}$$

where the C_j are residues of the rational function $|G(iw)|^2$ and defined as

$$C_j = \underset{w=-il_j}{res} |G(iw)|^2 = \lim_{w \to -il_j} (w + il_j)|G(iw)|^2 \tag{12}$$

As a short explanation of which poles were used for residue calculations, consider that the eigenvalues of the original transfer function must lie in the closed left hand plane. Any poles on the imaginary axis or right-half plane would contribute to an infinite response from a random input. Transforming from the l-plane to the w-plane, with the mapping $w = -li$, maps the closed, left-half l-plane to the closed, upper-half w-plane. A counter-clockwise contour around this region provides the line integral from which the integral of equation 11 can be determined.

Summing residues in this fashion for equation 11 results in equation 13 for the integral of the magnitude squared as a function of the pole and zero locations.

$$\int_{-\infty}^{+\infty} |G(iw)|^2 dw = pk^2 \sum_{i=1}^{n} \left[\frac{\prod_{j=1}^{m}\left(z_j^{\,2} - l_i^{\,2}\right)}{-l_i \prod_{k=1,k\neq i}^{n}\left(l_k^{\,2} - l_i^{\,2}\right)} \right] \tag{13}$$

Equation 13 is applied for up to second order transfer functions in Table 1 and third order transfer functions in Table A1 in Appendix A. Real and complex zeros and poles are included. The transfer functions in Table 1 have been modified such that the new gain, k^*, is the DC gain of the transfer function. This makes the distinction between pole and gain effects on mean-square response more meaningful. Table A1, has not been altered in this manner.

Table 1 – 1st And 2nd Order Frequency Response Integrals

Poles, zeros notes	$G(s)$	$\int_{-\infty}^{+\infty} \lvert G(iw)\rvert^2 dw$
1, 0	$k^*(-l)\frac{1}{(s-l)}$	$-pk^{*2} l$
2, 0	$k^*(l_1 l_2)\frac{1}{(s-l_1)(s-l_2)}$	$-\frac{pk^{*2} l_1 l_2}{(l_1 + l_2)}$
2, 0 Complex poles	$l_{1,2} = -zw_n \pm jw_d$	$\frac{pk^{*2} w_n}{2z}$
2, 1	$k^*\left(\frac{l_1 l_2}{-z}\right)\frac{(s-z)}{(s-l_1)(s-l_2)}$	$-\frac{pk^{*2} l_1 l_2 (l_1 l_2 + z^2)}{z^2 (l_1 + l_2)}$
2, 1 Complex poles	$l_{1,2} = -zw_n \pm jw_d$	$\frac{pk^{*2} w_n (w_n^{\,2} + z^2)}{2zz^2}$

Comments

Some initial interpretation of these results can be applied to Table 1. For a simple pole, E [y^2] goes as gain squared and directly with pole location. A higher break frequency leads to a higher mean-square. This would have implications for system time constant and required power in a feedback system. Looking at second order, underdamped poles, E[y^2] goes directly with natural frequency and inversely with damping ratio. It is probable that interesting conclusions could be drawn from further examination of these results. The "cost" of moving poles and zeros or changing natural frequency and damping ratios can be weighed versus changing performance.

E[y^2] AS A FUNCTION OF EIGEN-PROPERTIES AND SYSTEM MATRICES

Development

Mean-square values can also be derived as functions of eigen-properties and system matrices. This may lead to the understanding of a modal contribution to random response.

Defining eigenvalues as ?$_k$, right eigenvectors as $\underline{e}_k$ and normalized left eigenvectors as $\underline{v}_k^{\,t}$, the spectral decomposition [6] of a transfer function is as follows:

$$G(s) = \underline{c}^t \sum_{k=1}^{n} \frac{\underline{e}_k \underline{v}_k^{\,t}}{s - l_k} \underline{b} \tag{14}$$

c_t and b are the system output and input weighting vectors from a state-space realization. The frequency response is again found by letting s? iw giving $G(iw)$. The magnitude squared of the frequency response is $G(iw)\ G(iw)^*$ where the superscript *, denotes complex conjugate.

$$\left|G(iw)\right|^2 = \left(\underline{c}^t \sum_{k=1}^{n} \frac{\underline{e}_k \underline{v}_k^{\,t}}{iw - l_k} \underline{b} \right) \left(\underline{c}^t \sum_{j=1}^{n} \frac{\underline{e}_j \underline{v}_j^{\,t}}{iw - l_j} \underline{b} \right)^* \tag{15}$$

or

$$\begin{aligned} \left|G(iw)\right|^2 &= \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\underline{c}^t \underline{e}_k \underline{v}_k^{\,t} \underline{b} \left(\underline{c}^t \underline{e}_j \underline{v}_j^{\,t} \underline{b}\right)^*}{\left(iw - l_k\right)\left(-iw - l_j^{\,*}\right)} \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\underline{c}^t \underline{e}_k \underline{v}_k^{\,t} \underline{b} \left(\underline{c}^t \underline{e}_j \underline{v}_j^{\,t} \underline{b}\right)^*}{\left(w + i l_k\right)\left(w - i l_j^{\,*}\right)} \end{aligned} \tag{16}$$

Integrating the magnitude squared over all frequencies leads to mean-square values. We calculate this by the theory of residues as before.

$$\int_{-\infty}^{+\infty} \left|G(iw)\right|^2 dw = -2p \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\underline{c}^t \underline{e}_k \underline{v}_k^{\,t} \underline{b} \underline{c}^t \underline{e}_j \underline{v}_j^{\,t} \underline{b}}{l_k + l_j} \tag{17}$$

The complex conjugate goes away for real eigen-properties. It also goes away for complex eigen-properties because the sum is over all eigenvalues and eigenvectors which occur in conjugate pairs. Factoring the series gives

$$\int_{-\infty}^{+\infty} \left|G(iw)\right|^2 dw = -2p \sum_{k=1}^{n} \underline{c}^t \underline{e}_k \underline{v}_k^{\,t} \underline{b} \sum_{j=1}^{n} \frac{\underline{c}^t \underline{e}_j \underline{v}_j^{\,t} \underline{b}}{l_k + l_j} \tag{18}$$

The second summation can be simplified recognizing that

$$\left(l_k \underline{I} + \underline{A}\right)\underline{e}_j = (l_k + l_j)\underline{e}_j \tag{19}$$

.

Doing some algebra, right multiplying equation 19 by the jth normalized right eigenvector, and summing gives:

$$\sum_{j=1}^{n} \frac{\underline{e}_j \underline{v}_j^{\,t}}{l_k + l_j} = \left(l_k \underline{I} + \underline{A}\right)^{-1} \sum_{j=1}^{n} \underline{e}_j \underline{v}_j^{\,t} \tag{20}$$

The summation on the right becomes $\underline{e}\,\underline{v}^t$ and thus the identity matrix $\underline{I}$ where $\underline{e}$ and $\underline{v}^t$ are modal matrices.

$$\sum_{j=1}^{n} \frac{\underline{e}_j \underline{v}_j^{\,t}}{l_k + l_j} = \left(l_k \underline{I} + \underline{A}\right)^{-1} \tag{21}$$

Therefore, the integral under the magnitude squared in equation 18 becomes

$$\int_{-\infty}^{+\infty} \left|G(iw)\right|^2 dw = -2p\, \underline{c}^t \left\{ \sum_{k=1}^{n} \underline{e}_k \underline{c}^t \left(l_k \underline{I} + \underline{A}\right)^{-1} \underline{b} \underline{v}_k^{\,t} \right\} \underline{b} \tag{22}$$

If desired, Equation 22 can also be written strictly in terms of eigenvectors and eigenvalues.

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left|G(iw)\right|^2 dw = \\ &-2p\, \underline{c}^t \left\{ \sum_{k=1}^{n} \underline{e}_k \underline{c}^t \underline{e} \left(l_k \underline{I} + \underline{\Lambda}\right)^{-1} \underline{v}^t \underline{b} \underline{v}_k^{\,t} \right\} \underline{b} \end{aligned} \tag{23}$$

Λ is the diagonal matrix of eigenvalues and $\underline{A} = \underline{e}\,\underline{\Lambda}\,\underline{v}^t$.

Also, middle terms in the *summation* of Equation 22 are the scalar, rational function in Equation 24.

$$\underline{c}^t \left(l_k \underline{I} + \underline{A}\right)^{-1} \underline{b} = -G(-l_k) \tag{24}$$

and the magnitude-squared integral is written in terms of the transfer function.

$$\int_{-\infty}^{+\infty} \left|G(iw)\right|^2 dw = 2p\, \underline{c}^t \left\{ \sum_{k=1}^{n} G(-l_k) \underline{e}_k \underline{v}_k^{\,t} \right\} \underline{b} \tag{25}$$

Comments

Equation 25, in some sense, gives modal contributions to the mean-square response. As an example, the mean-square tire contact force in a two degree of freedom,actively damped, vehicle model (Figure 1) is calculated as a function of the passive damping coefficient, B_p. The roadway is assumed random in roughness and active damping is included such that the control force is $F_a = B_a V$. The transfer function between vertical contact point velocity, V_0, and tire force is given in Appendix B.

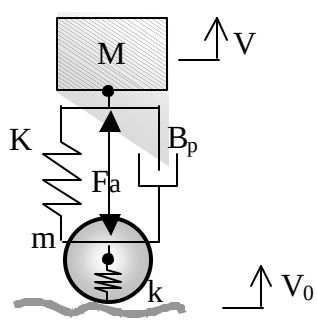

Figure 1 - 2 DOF (Quarter car) Suspension Model

Figure 2 shows the total mean-square response in the solid circles and the modal contributions in the hollow shapes and stars. For low damping, both modes are complex (as seen by superimposed contributions) and one mode significantly contributes to the total response. For higher damping, one complex and two real modes evolve. The complex mode again dominates the response. Near the break between all complex and complex and real modes, all modes contribute with one real mode much more than the other.

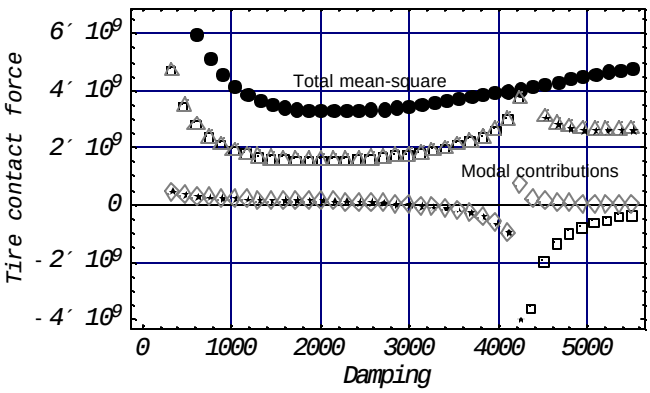

Figure 2 – Modal Contributions To Mean Square Tire Contact Force

CONCLUSIONS

Relationships have been developed relating poles and zeros, and system eigen-properties to mean square calculations of system output variables. It is hoped that these representations will lead to additional insights into the relationship between system properties and system performance. For example, how do you place poles and zeros to minimize or maximize mean-square response given other design constraints (such as power)? How do mode shapes (eigenvectors) contribute to response? For example, for one and two degree of freedom vehicle suspension models, the author has found that optimal active damping leads to a suspension tuning that cancels the dynamic effect of the suspension stiffness. Mathematically this results in a pole-zero cancellation and a reduction in system order for particular input/output transfer functions. The contribution of one mode to response vanishes as the optimal parameters are approached. Future work may examine how these new, mean-square relationships can increase insight into dynamic system design and optimal parameter (and feedback) selection.

REFERENCES

[1] Newland, D. E., 1984, An Introduction to Random Vibrations and Spectral Analysis, 2nd edition, Longman House, Essex, England.

[2] James, H. M., Nichols, N. B., and Phillips, R. S., 1947, Theory of Servomechanisms, MIT Radiation Laboratory Series, Vol. 25, McGraw Hill, New York.

[3] Redfield, Robin C. and Karnopp, D., 1989, "Performance Sensitivity of an Actively Damped Vehicle Suspension to Feedback Variation," J. of Dynamic Systems, Measurement, and Control, Vol. 111, No. 1, pp. 51-60.

[4] Redfield, Robin C., 1994, "Random Vibration and the Single Degree-of-Freedom Vibratory System: A Symbolic Quantification of Isolation and Packaging Performance," Journal of Vibration and Acoustics, Vol. 116, No. 1, pp. 1-5.

[5] Karnopp, D., 1989, "Analytical Results for Optimum Actively Damped Suspensions Under Random Excitation," Journal of Vibration, Acoustics, Stress, and Reliability in Design, Vol. 111, pp. 278-282.

[6] Kailath, T., 1980, Linear Systems, Prentice Hall, New Jersey.

APPENDIX A – Third Order Results

Table A1 – 3rd Order Frequency Response Integrals

Poles, zeros notes	$G(s)$	$\int_{-\infty}^{+\infty} \lvert G(iw)\rvert^2 dw$
3, 0	$\frac{k}{(s-l_1)(s-l_2)(s-l_3)}$	$-\frac{pk^2(l_1+l_2+l_3)}{l_1 l_2 l_3 (l_1+l_2)(l_1+l_3)(l_2+l_3)}$
3, 0 Complex poles	$l_{1,2} = -zw_n \pm jw_d$	$\frac{pk^2(l_3-2zw_n)}{2zw_n^3 l_3 (w_n^2 - 2zw_n l_3 + l_3^2)}$
3, 1	$\frac{k(s-z)}{(s-l_1)(s-l_2)(s-l_3)}$	$-\frac{pk^2[l_1 l_2 l_3 + z^2(l_1+l_2+l_3)]}{l_1 l_2 l_3 (l_1+l_2)(l_1+l_3)(l_2+l_3)}$
3, 1 Complex poles	$l_{1,2} = -zw_n \pm jw_d$	$\frac{pk^2(w_n^2 l_3 - 2zw_n z^2 + l_3 z^2)}{2zw_n^3 l_3 (w_n^2 - 2zw_n l_3 + l_3^2)}$
3, 2	$\frac{k(s-z_1)(s-z_2)}{(s-l_1)(s-l_2)(s-l_3)}$	$-\frac{pk^2[l_1 l_2 l_3 (l_1 l_2 + l_1 l_3 + l_2 l_3 + z_1^2 + z_2^2) + z_1^2 z_2^2 (l_1+l_2+l_3)]}{l_1 l_2 l_3 (l_1+l_2)(l_1+l_3)(l_2+l_3)}$
3, 2 Complex poles	$l_{1,2} = -zw_n \pm jw_d$	$\frac{pk^2[w_n^2 l_3 (w_n^2 - 2zw_n l_3 + z_1^2 + z_2^2) + z_1^2 z_2^2 (l_3 - 2zw_n)]}{2zw_n^3 l_3 (w_n^2 - 2zw_n l_3 + l_3^2)}$
3, 2 Complex zeros	$z_{1,2} = -z_z w_{nz} \pm jw_{dz}$	$-\frac{pk^2[l_1 l_2 l_3 (l_1 l_2 + l_1 l_3 + l_2 l_3 - 2w_{nz}^2 + 4z_z^2 w_{nz}^2) + w_{nz}^4 (l_1+l_2+l_3)]}{l_1 l_2 l_3 (l_1+l_2)(l_1+l_3)(l_2+l_3)}$
3, 2 Complex poles & zeros	$l_{1,2} = -zw_n \pm jw_d$ $z_{1,2} = -z_z w_{nz} \pm jw_{dz}$	$\frac{pk^2[w_n^2 l_3 (w_n^2 - 2zw_n l_3 - 2w_{nz}^2 + 4z_z^2 w_{nz}^2) + w_{nz}^4 (l_3 - 2zw_n)]}{2zw_n^3 l_3 (w_n^2 - 2zw_n l_3 + l_3^2)}$

APPENDIX B – Tire Force Transfer Function

$$\frac{F(s)}{V_0(s)} = \frac{k[s^s + \left(\frac{B_p}{m} + \frac{B_p + B_a}{M}\right)s^2 + K\left(\frac{1}{m} + \frac{1}{M}\right)s}{s^4 + \left(\frac{B_p}{m} + \frac{B_p + B_a}{M}\right)s^3 + \left(\frac{K}{M} + \frac{K+k}{m}\right)s^2 + \frac{k(B_p + B_a)}{mM}s + \frac{kK}{mM}} \tag{B1}$$

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